

N Policy For Repairable Bulk Arrival Queueing Model With Setup, Second Multi Optional Service Facility Under Restricted Number Of Server Vacations

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ABSTRACT: This paper analyses a repairable batch arrival queueing model with a second multi optional service(SOS) channel under N policy in which the server takes restricted number of multiple vacations during his idle period. The server leaves the system for vacation as soon as the system empties and after returning from vacation if the server finds N or more customers in the system then he immediately starts the setup work. Otherwise he repeats his vacation until he finally finds at least N customers or returns to the system after taking L vacations. ie At the end of Lth vacation if the server finds less than N customers he joins the system and stays idle until the queue length reaches at least N to start the setup work. After the completion of setup work, busy period initiates. The server may undergo unpredictable breakdowns during the service and sent for repair immediately. As soon as the repair is completed, the server returns to the customer whose service was interrupted. The system size distribution at a random epoch and mean system length are calculated and the corresponding results for the classical single and multiple vacation models are obtained as special case.

KEYWORDS: Npolicy, Bulkarrival, Second multi optional Service, Restricted Vacation.

I. INTRODUCTION

The single and multiple vacation N-policy queueing models allow the server to take vacations whenever the system becomes empty. In the case of single vacation model the server after returning from the vacation joins the system even if the required number of customers are not present. That is, the server stays idle in the system after returning from vacation until at least N customers accumulate. In the case of classical multiple vacation the server may take unlimited number of vacations before joining the system. In the present paper a modified multiple vacation policy is considered, where the server takes at most J vacations repeatedly until at least N customers accumulate for service. If the number of arrivals does not reach N at the end of Jth vacation, then the server joins the system and remains idle until the number of arrivals in the queue reaches N. Thus the present paper is a generalization of the classical single and multiple vacation models. Also the second optional service considered in the present model consists of multi c-optional services so that the customer who completes the First Essential Service (FES) may choose the ith optional service with

probability r_i (or) leaves the system with probability $(1-r)$ where $r = \sum_{i=1}^c r_i$. The server undergoes unpredictable

breakdown during service and resumes his work after undergoing the repairwork. It is also assumed that the server does some preparatory work before starting the FES of each cycle.

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II. RELATED WORK

In classical single server queueing models an idle server will start the service as soon as a customer enters to the empty system. But in many situations it is important to decide when the server should start his service, as frequent setup inevitably make the operating cost too high. The N policy introduced by Yadin and Naor (1963) is the most general control policy in which the server does not start his service until there are N customers in the queue. This minimizes server switch overs and avoids excessive frequent use of setups. The N policy queue have been widely used to provide stochastic modelling of many problems arising in production inventory system, in telephone switching system and in quality control problems. The study of the N policy M/G/1 queueing system was first analysed by Heyman(1968) and was developed by several researchers such as Wang and Ke(2002). In recent years, queues due to server vacations have emerged as an important area of queueing theory and have been studied exhaustively and successfully, in various applications. At present most of the studies are developed for batch arrival vacation models under various vacation policies. The first study of vacation models with N policy was done by Kella(1989). The various extensions of the vacation models under N policy can be referred to Lee et al.(1994,1995), Ke(2001) and others. Later Ke et al.(2010) studied the operating characteristics of an M/G/1 queueing system with a randomized control policy and at most J vacations. Queueing models with server breakdowns are more realistic representation of systems. The server may undergo unpredictable breakdowns during the service and sent for repair immediately. As soon as the repair is completed, the server returns to the customer whose service was interrupted. Discussing queueing systems with N policy and server breakdowns, Wang(1995) first proposed Markovian queueing system under N policy and server breakdowns and derived analytical steady-state solutions of the N policy M/M/1 queue. Wang and Ke(2002) analysed the control policy for M/G/1 queueing system with an unreliable server. The N policy Markovian system with finite capacity and a non-reliable server was proposed by Wang and Hisch(1995). Recently Ke, Huang, Pearn(2010) has studied randomized policy of a poisson input queue with J vacations. Further Ke, Huang, Pearn(2010) discussed the randomized vacation policy for a batch arrival queue. Later Huang, Pearn(2011) analysed performance measures and randomized optimization of unreliable server $M^X/G/1$ vacation system. In classical repeated vacation, the server takes infinite vacations which is unrealistic. Thus the server can be restricted to have at most L vacations in one cycle. The L vacation queueing system can be considered as generalization of single and multiple vacation when $L=1$ and $L=\infty$. In the present paper the authors analyse the N policy of repairable batch arrival queueing system with general setup time in which the service discipline involves more than one service, of which one is essential and the other is multi optional. The server takes at most L vacations whenever the system becomes empty. The PGF of the steady state system size probabilities is obtained in a closed form so that various performance measures can be calculated easily.

III. MATHEMATICAL ANALYSIS OF THE SYSTEM

MODEL DESCRIPTION:

Arrival Pattern: Present Paper considers an $M^X/G/1$ queueing system where the arrivals occur in batches according to a compound Poisson process with random batch size X, group arrival rate λ and probability distribution $g_k = \Pr(X=k)$ $k=1,2,3,\dots$. Arriving customers form a single waiting line and the service is done one by one.

Vacation Policy: The server is turned off and leaves the system for vacation of random length V as soon as the system empties. After returning from the vacation if the server finds less than N customers he repeats his vacation until he finds at least N customers or returns to the system after taking L vacations. The vacation times are assumed to be iid with general distribution V(t). At the end of L^{th} vacation if the server finds less than N customers he joins the system and stays idle until the queue length reaches at least N. On the other hand if on returning from vacation the server finds N or more customers in the system, he immediately starts preparatory work which corresponds to the setup time. Setup time follows general distribution D(t). At the end of the setup operation the server begins to serve the customers exhaustively.

Busy Period: During busy period, the server provides single First Essential Service(FES) to all the arriving customers in the first phase and "multi-optional" service in the second phase. The multi-optional service consists of c types of heterogeneous service. As soon as the FES is completed, the customers may either leave the system or opt for any of the c services in the second phase. It is assumed that, the probability that the customer opts the i^{th} type ($1 \leq i \leq c$) of

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service in the second phase is r_i ($0 \leq r_i \leq 1$) and the probability that the customer leaves the system after finishing the FES is $(1-r)$ where $r = \sum_{i=1}^c r_i$. The service time random variables S and S_i ($i = 1$ to c) of FES and SOS respectively are assumed to be mutually independent of each other having heterogeneous general law of distributions with distribution functions $S(x)$ and $S_i(x)$ and Laplace Stieltjes Transform (LST) $S^*(\theta)$ and $S_i^*(\theta)$, $i = 1$ to c respectively. The system is denoted by $M_N^x / G, G_{i(1 \leq i \leq c)} / 1 / Jvacation$

Breakdown and Repair Period: It is assumed that the breakdowns occur during the services and the server's life times are assumed to follow exponential distributions with mean $1/a$ in FES and $1/a_i$ in i^{th} SOS service. Whenever the breakdowns occur, the server is sent for repair immediately and the customer just being served, waits in the service facility for the server to return from repair facility to complete the remaining service. The repair times R and R_i ($i = 1$ to c) are arbitrarily distributed with probability distribution functions $R(y)$ and $R_i(y)$, density functions $r(y)$ and $r_i(y)$ and the first and second moments $E(R_i^k)$, $k = 1, 2$ and $i = 1$ to c are the breakdowns occur in FES and SOS. Immediately after the server is fixed, the customer waiting in the service facility is taken up for service. Further, it is assumed that the service time for a customer is cumulative and after repair, the server is as good as new.

- $N(t)$: The system size at time t λ : Group arrival rate
- X : Group size random variable g_k : $\Pr(X = k)$, $k = 1, 2, 3, \dots$
- $X(z)$: Probability generating function of X .
- $Y(t)$: $\{0, 1, 2, 3, 4, 5, 6\}$ according as the system in vacation, build up, setup, busy with FES, busy with i^{th} SOS, repair in FES and repair in i^{th} SOS state respectively.

The notations of Random Variables (RV), Cumulative Distribution Functions (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transform (LST) and its k^{th} moments of the RVs are listed below

	RV	CDF	PDF	LST	K^{th} moments
Vacation Time	V	V(x)	v(x)	$V^*(\theta)$	$E(V^k)$
Setup time	D	D(x)	d(x)	$D^*(\theta)$	$E(D^k)$
FES	S	S(x)	s(x)	$S^*(\theta)$	$E(S^k)$
SOS	S_i	$S_i(x)$	$s_i(x)$	$S_i^*(\theta)$	$E(S_i^k)$
Repair time in FES	R	R(y)	r(y)	$R^{*1}(\theta_1)$	$E(R^k)$
Repair time in SOS	R_i	$R_i(y)$	$r_i(y)$	$R_i^{*1}(\theta_1)$	$E(R_i^k)$, $i=1$ to c

Let $V^0(t), D^0(t), S^0(t), S_i^0(t), (i=1$ to $c) R^0(t)$ and $R_i^0(t), (i=1$ to $c)$ denote the remaining (vacation time, setup time, FES time, SOS time, repair time due to FES and repair time due to SOS) respectively at time t . Then the state space $\{N(t), \delta(t)\}$, where $\delta(t) = (V^0(t), D^0(t), S^0(t), S_i^0(t), R^0(t)$ and $R_i^0(t))$ according as $Y(t) = 0$ to 6 respectively defines a bivariate Markov process.

Let

$$Q_n(x, t) dt = \Pr(N(t) = n, x \leq V^0(t) \leq x + dt, Y(t) = 0), \quad n \geq 0,$$

$$P_{n,0}(t) = \Pr(N(t) = n, Y(t) = 1), \quad 0 \leq n \leq N-1$$

$$D_n(x, t) dt = \Pr(N(t) = n, x \leq D^0(t) \leq x + dt, Y(t) = 2), \quad n \geq N$$

$$P_n(x, t) dt = \Pr(N(t) = n, x \leq S^0(t) \leq x + dt, Y(t) = 3), \quad n \geq 1$$

$$P_{n,i}(x, t) dt = \Pr(N(t) = n, x \leq S_i^0(t) \leq x + dt, Y(t) = 4), \quad n \geq 1, 1 \leq i \leq c$$

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$$B_n(x, y, t)dt = \Pr(N(t) = n, S^0(t) = x, y \leq R^0(t) \leq y + dt, Y(t) = 5)), n \geq 1$$

$$B_{n,i}(x, y, t)dt = \Pr(N(t) = n, S_i^0(t) = x, y \leq R_i^0(t) \leq y + dt, Y(t) = 6)), n \geq 1, 1 \leq i \leq c$$

Thus $P_{n,0}(t)$ denotes the probability that there are n customers in the system at time t when the system is in build up state. $Q_n(x, t)$, $D_n(x, t)$, $P_n(x, t)$ and $P_{n,i}(x, t)$ $i=1$ to c respectively denote the probability that there are n customers in the system at arbitrary epoch with the remaining vacation time, setup time first essential and second optional service time lie in the interval $[x, x+dt]$. $B_n(x, y, t) dt$ and $B_{n,i}(x, y, t) dt$, $i = 1$ to c are the joint probability that at time t , there are n customers in the system, the remaining service time for the customer under service is equal to x , and the server is being repaired with the remaining repair time between y and $y + dt$ in first essential and second optional service time respectively. Further $Q_n(0)$, $D_n(0)$, $P_{n,i}(0)$, $B_{n,i}(x, 0)$, $i = 1, 2$ denote the probability that there are n customers in the system at the termination of vacation period, setup period, service time and repair time respectively.

System size Distribution at Random Epoch: Following the argument of Cox(1955) and observing the changes of states during the interval $(t, t+\Delta t)$ for any time t , the steady state equations are given by

Vacation State

$$\frac{-d}{dx} Q_{0,1}(x) = -\lambda Q_{0,1}(x) + \bar{P}(0)v(x), \text{ where } \bar{P}(0) = P_1(0)(1-r) + \sum_{i=1}^c P_{1,i}(0)$$

$$\frac{-d}{dx} Q_{0,j}(x) = -\lambda Q_{0,j}(x) + Q_{0,j-1}(0)v(x) \quad 2 \leq j \leq L$$

$$\frac{-d}{dx} Q_{n,1}(x) = -\lambda Q_{n,1}(x) + \lambda \sum_{k=1}^n Q_{n-k,1}(x)g_k \quad n \geq 1$$

$$\frac{-d}{dx} Q_{n,j}(x) = -\lambda Q_{n,j}(x) + Q_{n,j-1}(0)v(x) + \lambda \sum_{k=1}^n Q_{n-k,j}(x)g_k \quad 1 \leq n \leq N-1 \text{ and } 2 \leq j \leq L$$

$$\frac{-d}{dx} Q_{n,j}(x) = -\lambda Q_{n,j}(x) + \lambda \sum_{k=1}^n Q_{n-k,j}(x)g_k \quad n \geq N \text{ and } 2 \leq j \leq L$$

Build up State

$$\lambda P_{0,0} = Q_{0L}(0)$$

$$\lambda P_{n,0} = Q_{nL}(0) + \lambda \sum_{k=1}^n P_{n-k,0}g_k \quad 1 \leq n \leq N-1$$

Setup state

$$\frac{-d}{dx} D_N(x) = -\lambda D_N(x) + \sum_{j=1}^L Q_{N,j}(0)d(x) + \lambda \sum_{k=1}^n P_{n-k,0}g_k d(x)$$

$$\frac{-d}{dx} D_n(x) = -\lambda D_n(x) + \lambda \sum_{k=1}^{n-N} D_{n-k}(x)g_k + \sum_{j=1}^L Q_{n,j}(0)d(x) + \lambda \sum_{k=n-N+1}^n P_{n-k,0}g_k d(x) \quad n \geq N+1$$

Busy with FES

$$\frac{-d}{dx} P_1(x) = -(\lambda + a)P_1(x) + (1-r)P_2(0)s(x) + \sum_{i=1}^c P_{2,i}(0)s(x) + B_1(x,0)$$

$$-\frac{d}{dx} P_n(x) = -(\lambda + a)P_n(x) + (1-r)P_{n+1}(0)s(x) + \lambda \sum_{k=1}^{n-1} P_{n-k}(x)g_k + \sum_{i=1}^c P_{n+1,i}(0)s(x) + B_n(x,0)$$

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$$\frac{-d}{dx} P_n(x) = -(\lambda + a)P_n(x) + (1-r)P_{n+1}(0)s(x) + \lambda \sum_{k=1}^{n-1} P_{n-k}(x)g_k + \sum_{i=1}^c P_{n+1,i}(0)s(x) + B_n(x,0) + D_n(0)s(x) \quad \begin{matrix} 2 \leq n \leq N-1 \\ n \geq N \end{matrix}$$

Busy with ith type of SOS

$$-\frac{d}{dx} P_{1,i}(x) = -(\lambda + a_i)P_{1,i}(x) + r_1 P_1(0)s_i(x) + B_{1,i}(x,0)$$

$$-\frac{d}{dx} P_{n,i}(x) = -(\lambda + a_i)P_{n,i}(x) + r_1 P_n(0)s_i(x) + \lambda \sum_{k=1}^{n-1} P_{n-k,i}(x)g_k + B_{n,i}(x,0), \quad n \geq 2, 1 \leq i \leq c$$

Breakdown during FES

$$\frac{-\partial}{\partial y} B_1(x,y) = -\lambda B_1(x,y) + aP_1(x)r(y) \quad \frac{-\partial}{\partial y} B_n(x,y) = -\lambda B_n(x,y) + aP_n(x)r(y) + \lambda \sum_{k=1}^{n-1} B_{n-k}(x,y)g_k, \quad n \geq 2$$

Breakdown during with ith type of SOS

$$\frac{-\partial}{\partial y} B_{1,i}(x,y) = -\lambda B_{1,i}(x,y) + a_i P_{1,i}(x)r_i(y)$$

$$\frac{-\partial}{\partial y} B_{n,i}(x,y) = -\lambda B_{n,i}(x,y) + a_i P_{n,i}(x)r_i(y) + \lambda \sum_{k=1}^{n-1} B_{n-k,i}(x,y)g_k, \quad n \geq 2, 1 \leq i \leq c$$

Taking the LST of the above steady-state equations w.r.to x, we have

$$\theta Q_{0,1}^*(\theta) - Q_{0,1}(0) = \lambda Q_{0,1}^*(\theta) - \bar{P}(0)V^*(\theta) \quad (1.1)$$

$$\theta Q_{0,j}^*(\theta) - Q_{0,j}(0) = \lambda Q_{0,j}^*(\theta) - Q_{0,j-1}(0)V^*(\theta), \quad 2 \leq j \leq L \quad (1.2)$$

$$\theta Q_{n,1}^*(\theta) - Q_{n,1}(0) = \lambda Q_{n,1}^*(\theta) - \lambda \sum_{k=1}^n Q_{n-k,1}^*(\theta)g_k \quad n \geq 1 \quad (1.3)$$

$$\theta Q_{n,j}^*(\theta) - Q_{n,j}(0) = \lambda Q_{n,j}^*(\theta) - Q_{n,j-1}(0)V^*(\theta) - \lambda \sum_{k=1}^n Q_{n-k,j}^*(\theta)g_k \quad \begin{matrix} 1 \leq n \leq N-1 \\ 2 \leq j \leq L \end{matrix} \quad (1.4)$$

$$\theta Q_{n,j}^*(\theta) - Q_{n,j}(0) = \lambda Q_{n,j}^*(\theta) - \lambda \sum_{k=1}^n Q_{n-k,j}^*(\theta)g_k \quad n \geq N \text{ and } 2 \leq j \leq L \quad (1.5)$$

$$\lambda P_{0,0} = Q_{0L}(0) \quad (1.6)$$

$$\lambda P_{n,0} = Q_{nL}(0) + \lambda \sum_{k=1}^n P_{n-k,0}g_k \quad 1 \leq n \leq N-1 \quad (1.7)$$

$$\theta D_N^*(\theta) - D_N(0) = \lambda D_N^*(\theta) - \sum_{j=1}^L Q_{N,j}(0)D^*(\theta) - \lambda D^*(\theta) \sum_{k=1}^N P_{N-k,0}g_k \quad (1.8)$$

$$\theta D_n^*(\theta) - D_n(0) = \lambda D_n^*(\theta) - \sum_{j=1}^L Q_{n,j}(0)D^*(\theta) - \lambda \sum_{k=1}^{n-N} D_{n-k}^*(\theta)g_k$$

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$$-\lambda D^*(\theta) \sum_{k=n-N+1}^n P_{n-k,0}(\theta) g_k \quad n \geq N+1 \tag{1.9}$$

$$\theta P_1^*(\theta) - P_1(0) = (\lambda + a)P_1^*(\theta) - (1-r)P_2(0)S^*(\theta) - \sum_{i=1}^c P_{2,i}(0)S^*(\theta) - B_1^*(\theta,0) \tag{1.10}$$

$$\theta P_n^*(\theta) - P_n(0) = (\lambda + a)P_n^*(\theta) - (1-r)P_{n+1}(0)S^*(\theta) - \sum_{i=1}^c P_{n+1,i}(0)S^*(\theta) - \lambda \sum_{k=1}^{n-1} P_{n-k}^*(\theta)g_k - B_n^*(\theta,0) \quad 2 \leq n \leq N-1 \tag{1.11}$$

$$\theta P_n^*(\theta) - P_n(0) = (\lambda + a)P_n^*(\theta) - (1-r)P_{n+1}(0)S^*(\theta) - \sum_{i=1}^c P_{n+1,i}(0)S^*(\theta) - \lambda \sum_{k=1}^{n-1} P_{n-k}^*(\theta)g_k - D_n(0)S^*(\theta) - B_n^*(\theta,0) \quad n \geq N \tag{1.12}$$

$$\theta P_{1,i}^*(\theta) - P_{1,i}(0) = (\lambda + a_i)P_{1,i}^*(\theta) - r_i P_1(0)S_i^*(\theta) - B_{1,i}^*(\theta,0) \tag{1.13}$$

$$\theta P_{n,i}^*(\theta) - P_{n,i}(0) = (\lambda + a_i)P_{n,i}^*(\theta) - r_i P_n(0)S_i^*(\theta) - \lambda \sum_{k=1}^{n-1} P_{n-k,i}^*(\theta)g_k - B_{n,i}^*(\theta,0) \quad n \geq 2, 1 \leq i \leq c \tag{1.14}$$

$$\frac{\partial}{\partial y} B_1^*(\theta, y) = -\lambda B_1^*(\theta, y) + aP_1^*(\theta)r(y) \tag{1.15}$$

$$\frac{\partial}{\partial y} B_n^*(\theta, y) = -\lambda B_n^*(\theta, y) + aP_n^*(\theta)r(y) + \lambda \sum_{k=1}^{n-1} B_{n-k}^*(\theta, y)g_k \quad n \geq 2 \tag{1.16}$$

$$\frac{\partial}{\partial y} B_{1,i}^*(\theta, y) = -\lambda B_{1,i}^*(\theta, y) + a_i P_{1,i}^*(\theta)r_i(y) \tag{1.17}$$

$$\frac{\partial}{\partial y} B_{n,i}^*(\theta, y) = -\lambda B_{n,i}^*(\theta, y) + a_i P_{n,i}^*(\theta)r_i(y) + \lambda \sum_{k=1}^{n-1} B_{n-k,i}^*(\theta, y)g_k \quad n \geq 2 \tag{1.18}$$

Taking the LST w.r.to y from equations (1.15) to (1.18), we have

$$\theta_1 B_1^{**1}(\theta, \theta_1) - B_1^*(\theta,0) = \lambda B_1^{**1}(\theta, \theta_1) - aP_1^*(\theta)R^*(\theta_1) \tag{1.19}$$

$$\theta_1 B_n^{**1}(\theta, \theta_1) - B_n^*(\theta,0) = \lambda B_n^{**1}(\theta, \theta_1) - aP_n^*(\theta)R^*(\theta_1) - \lambda \sum_{k=1}^{n-1} B_{n-k}^{**1}(\theta, \theta_1)g_k \tag{1.20}$$

$$\theta_1 B_{1,i}^{**1}(\theta, \theta_1) - B_{1,i}^*(\theta,0) = \lambda B_{1,i}^{**1}(\theta, \theta_1) - a_i P_{1,i}^*(\theta)R_i^{**1}(\theta_1) \tag{1.21}$$

$$\theta_1 B_{n,i}^{**1}(\theta, \theta_1) - B_{n,i}^*(\theta,0) = \lambda B_{n,i}^{**1}(\theta, \theta_1) - a_i P_{n,i}^*(\theta)R_i^{**1}(\theta_1) - \lambda \sum_{k=1}^{n-1} B_{n-k,i}^{**1}(\theta, \theta_1)g_k \quad n \geq 2, 1 \leq i \leq c \tag{1.22}$$

Probability Generating Function

Now to obtain the partial PGFs of the number of customers in the system, the following partial PGFs are defined.

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$$\begin{aligned}
 P_0(z) &= \sum_{n=0}^{N-1} P_{n,0} z^n & Q_j^*(z, \theta) &= \sum_{n=0}^{\infty} Q_{n,j}^*(\theta) z^n & Q_j(z, 0) &= \sum_{n=0}^{\infty} Q_{n,j}(0) z^n & D^*(z, \theta) &= \sum_{n=N}^{\infty} D_n^*(\theta) z^n \\
 D(z, 0) &= \sum_{n=N}^{\infty} D_n(0) z^n & P^*(z, \theta) &= \sum_{n=1}^{\infty} P_n^*(\theta) z^n & P(z, 0) &= \sum_{n=1}^{\infty} P_n(0) z^n & B^{**1}(z, \theta, \theta_1) &= \sum_{n=1}^{\infty} B_n^{**1}(\theta, \theta_1) z^n \\
 B^{*1}(z, \theta, 0) &= \sum_{n=1}^{\infty} B_n^*(\theta, 0) z^n & P_i^*(z, \theta) &= \sum_{n=1}^{\infty} P_{n,i}^*(\theta) z^n & P_i(z, 0) &= \sum_{n=1}^{\infty} P_{n,i}(0) z^n \\
 B_i^{**1}(z, \theta, \theta_1) &= \sum_{n=1}^{\infty} B_{n,i}^{**1}(\theta, \theta_1) z^n & B_i^{*1}(z, \theta, 0) &= \sum_{n=1}^{\infty} B_{n,i}^*(\theta, 0) z^n
 \end{aligned}$$

Multiplying the equations (1.1) and (1.2) by suitable powers of z and then adding over n=0 to ∞ we have $(\theta - w_x(z))Q_1^*(z, \theta) = Q_1(z, 0) - \bar{P}(0)V^*(\theta)$ where $w_x(z) = \lambda(1 - X(z))$ (1.23)

and at $\theta = w_x(z)$ $Q_1(z, 0) = \bar{P}(0)V^*(w_x(z))$ and (1.24)

$$Q_1^*(z, \theta) = \bar{P}(0) \frac{[V^*(w_x(z)) - V^*(\theta)]}{(\theta - w_x(z))} \tag{1.25}$$

Similarly equations (1.3) to (1.5) lead to

$$Q_j(z, 0) = V^*(w_x(z)) \sum_{n=0}^{N-1} z^n Q_{n,j-1}(0) \quad 2 \leq j \leq L \tag{1.26}$$

$$\text{and } Q_j^*(z, \theta) = \frac{[V^*(w_x(z)) - V^*(\theta)]}{\theta - w_x(z)} \sum_{n=0}^{N-1} z^n Q_{n,j-1}(0) \quad 2 \leq j \leq L \tag{1.27}$$

The sum of the equations (1.24) and (1.26) over j=1 to L yields

$$Q(z, 0) = \sum_{j=1}^L Q_j(z, 0) = V^*(w_x(z))y(z) \tag{1.28}$$

$$\text{where } y(z) = \sum_{j=1}^{L-1} \sum_{n=0}^{N-1} Q_{n,j}(0) z^n + \bar{P}(0) \tag{1.29}$$

The equations (1.25) and (1.27) imply,

$$Q^*(z, \theta) = \sum_{j=1}^L Q_j^*(z, \theta) = \frac{[V^*(w_x(z)) - V^*(\theta)]}{\theta - w_x(z)} y(z) \tag{1.30}$$

Next to calculate the PGF corresponding to setup state, equations (1.8) and (1.9) are used which in turn lead to

$$D(z, 0) = D^*(w_x(z))[(V^*(w_x(z)) - 1)y(z) + \bar{P}(0) - P_0(z)w_x(z)] \text{ and } \tag{1.31}$$

$$D^*(z, \theta) = \frac{(D^*(w_x(z)) - D^*(\theta))}{\theta - w_x(z)} [(V^*(w_x(z)) - 1)y(z) + \bar{P}(0) - P_0(z)w_x(z)] \tag{1.32}$$

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Equations (1.6) and (1.7) recursively imply $\lambda P_{n,0} = \left(\sum_{k=0}^n Q_{n-k,L}(0) \pi_k \right)$, $0 \leq n \leq N-1$ where $\pi_0 = 1$ and

$\pi_n = \sum_{k=1}^n \pi_{n-k} g_k$. Defining $\psi_0 = \alpha_0^L$, $\psi_n = \sum_{k=0}^n \alpha_k^{(L)} \pi_{n-k}$ where $\alpha^{(i)} = \sum_{k=0}^n \alpha_{n-k}^{(i-1)} \alpha_k$ $k=0$ to n , it is found that

$$\lambda P_{n,0} = \psi_n \bar{P}(0) \text{ for every } n. \text{ Hence } P_0(z) = \bar{P}(0) \psi(z) \text{ for every } n \tag{1.33}$$

where
$$\psi(z) = \sum_{n=0}^{N-1} \frac{\psi_n z^n}{\lambda}$$

Next to evaluate $y(z)$, the following arguments are used. If α_n denotes the probability that ‘n’ customers arrive during

a vacation time V , then $V^*(w_x(z)) = \sum_{n=0}^{\infty} \alpha_n z^n$ (Gross&Haris 1998)

and hence $Q_{n,1}(0) = \alpha_n \bar{P}(0)$ $n \geq 0$ (1.34)

By conditioning the arrival size during previous vacation

$$Q_{n,j}(0) = \sum_{k=0}^n \alpha_k Q_{n-k,j-1}(0) \quad 2 \leq j \leq L \text{ and } 0 \leq n \leq N-1 \tag{1.35}$$

At $n=0$, $Q_{0,j}(0) = \alpha_0 Q_{0,j-1} = \alpha_0^{j-1} Q_{0,1} = \alpha_0^j \bar{P}(0)$

Now adding equations (1.34) and (1.35) over $j=1$ to L for $0 \leq n \leq N-1$

$$\sum_{j=1}^L Q_{n,j}(0) = \alpha_n \bar{P}(0) + \sum_{j=1}^{L-1} \sum_{k=0}^n \alpha_k Q_{n-k,j}(0) \tag{1.36}$$

By letting $\delta_0 = \bar{P}(0) + \sum_{j=1}^{L-1} Q_{0,j}(0)$ and $\delta_n = \sum_{k=0}^n \alpha_k \delta_{n-k} - Q_{n,L}(0)$ $1 \leq n \leq N-1$

For further simplification β_n 's are introduced. Let $\beta_0 = 1 - \alpha_0^L$, $\beta_n = \sum_{j=1}^n \frac{\alpha_j \beta_{n-j}}{1 - \alpha_0} - \alpha_n^{(L)}$

$1 \leq n \leq N-1$. Then δ_n 's can be expressed in terms of β_n 's by induction as $\delta_n = \frac{\beta_n}{1 - \alpha_0} \bar{P}(0)$. Thus substituting δ_n 's in

eqn.(1.29) implies $y(z) = \sum_{n=0}^{N-1} \delta_n z^n = \bar{P}(0) \sum_{n=0}^{N-1} \frac{\beta_n z^n}{(1 - \alpha_0)}$ (1.37)

Substituting for $y(z)$ and $P_0(z)$ in equations (1.30),(1.31) and (1.32), we have

$$Q^*(z, \theta) = \sum_{j=1}^L Q_j^*(z, \theta) = \frac{[V^*(w_x(z)) - V^*(\theta)]}{(\theta - w_x(z))} \left(\sum_{n=0}^{N-1} \frac{\beta_n z^n}{(1 - \alpha_0)} \right) \bar{P}(0) \tag{1.38}$$

$$D^*(z, \theta) = \frac{(D^*(w_x(z)) - D^*(\theta))}{(\theta - w_x(z))} \left[(V^*(w_x(z)) - 1) \sum_{n=0}^{N-1} \frac{\beta_n z^n}{(1 - \alpha_0)} + 1 - \psi(z) w_x(z) \right] \bar{P}(0) \tag{1.39}$$

$$D(z, 0) = D^*(w_x(z)) \left[(V^*(w_x(z)) - 1) \sum_{n=0}^{N-1} \frac{\beta_n z^n}{(1 - \alpha_0)} + 1 - \psi(z) w_x(z) \right] \bar{P}(0) \tag{1.40}$$

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Hence $D(z,0) - \bar{P}(0) = -\bar{P}(0)w_x(z)I_N^{SR}(z)$ where

$$I_N^{SR}(z) = \left[D^*(w_x(z)) \left[\frac{1 - V^*(w_x(z))}{w_x(z)} \sum_{n=0}^{N-1} \frac{\beta_n z^n}{(1 - \alpha_0)} \right] + \frac{1 - D^*(w_x(z))}{w_x(z)} + \psi(z)D^*(w_x(z)) \right] \quad (1.41)$$

The generating functions of the system size when the server is in breakdown state with the customer waiting in the service facility for the server to return are calculated by using equations (1.19) to (1.22) and given below

$$B^*(z, \theta, 0) = aP^*(z, \theta)R^{*1}(w_x(z)) \quad (1.42)$$

$$B^{*1}(z, \theta, \theta_1) = \frac{aP^*(z, \theta)(R^{*1}(w_x(z)) - R^{*1}(\theta_1))}{(\theta_1 - w_x(z))} \quad (1.43)$$

For $1 \leq i \leq c$ $B_i^*(z, \theta, 0) = a_i P_i^*(z, \theta) R_i^{*1}(w_x(z))$ (1.44)

$$B_i^{*1}(z, \theta, \theta_1) = \frac{a_i P_i^*(z, \theta)(R_i^{*1}(w_x(z)) - R_i^{*1}(\theta_1))}{(\theta_1 - w_x(z))} \quad (1.45)$$

The partial generating functions of the system size when the server is busy with i^{th} type of SOS are obtained using equations (1.13) and (1.14) and given by

$$P_i(z, 0) = r_i S_i^*(h_{a_i}(w_x(z)))P(z, 0) \text{ where } h_{a_i}(w_x(z)) = w_x(z) + a_i(1 - R_i^{*1}(w_x(z))) \quad (1.46)$$

$$P_i^*(z, \theta) = \frac{r_i P(z, 0)(S_i^*(h_{a_i}(w_x(z))) - S_i^*(\theta))}{(\theta - h_{a_i}(w_x(z)))} \quad (1.47)$$

To derive the generating functions of the system size at arbitrary epoch and service completion Epoch, when the server is busy with FES, the equations (1.10) to (1.12) and (1.46) are used and resulted in

$$(\theta - h_a(w_x(z)))P^*(z, \theta) = \frac{P(z, 0)}{z} [z - S^*(\theta)(1 - r) - \sum_{i=1}^c S_i^*(h_{a_i}(w_x(z)))r_i S^*(\theta)] - S^*(\theta)[D(z, 0) - \bar{P}(0)] \quad (1.48)$$

where $h_a(w_x(z)) = w_x(z) + a(1 - R^{*1}(w_x(z)))$

At $\theta = h_a(w_x(z))$, $P(z, 0) = \frac{zS^*(h_a(w_x(z)))[D(z, 0) - \bar{P}(0)]}{z - H_{SOS}^*(h(w_x(z)))}$ (1.49)

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where $H_{SOS}^*(h(w_x(z))) = S^*(h_a(w_x(z))) \left[1 - r + \sum_{i=1}^c r_i S_i^*(h_{a_i}(w_x(z))) \right]$

$$P^*(z, \theta) = \frac{z(S^*(h_a(w_x(z)) - S^*(\theta)) [D(z,0) - \bar{P}(0)])}{z - H_{SOS}^*(h(w_x(z))(\theta - h_a(w_x(z))))} \tag{1.50}$$

Thus the partial PGFs of the system size for the proposed model are obtained from equations (1.33, 1.38, 1.39, 1.43, 1.45, 1.47, 1.50) as

$$\sum_{j=1}^L Q_j^*(z,0) = \bar{P}(0) \frac{(1 - V^*(w_x(z))) \sum_{n=0}^{N-1} \beta_n z^n}{w_x(z) (1 - \alpha_0)} \tag{1.51}$$

$$P_0(z) = \bar{P}(0) \psi(z) \tag{1.52}$$

$$D^*(z,0) = \frac{\bar{P}(0)(1 - D^*(w_x(z)))}{w_x(z)} \left[(V^*(w_x(z)) - 1) \sum_{n=0}^{N-1} \frac{\beta_n z^n}{(1 - \alpha_0)} + 1 - \psi(z)w_x(z) \right] \tag{1.53}$$

$$P^*(z,0) = \frac{\bar{P}(0)z(S^*(h_a(w_x(z)) - 1)I_N^{SR}(z))}{h_a(w_x(z))(z - H_{SOS}^*(h(w_x(z))))} \tag{1.54}$$

For $(1 \leq i \leq c)$

$$P_i^*(z,0) = \frac{\bar{P}(0)r_i z S^*(h_a(w_x(z))I_N^{SR}(z)w_x(z))(S_i^*(h_{a_i}(w_x(z))) - 1)}{h_{a_i}(w_x(z))(z - H_{SOS}^*(h(w_x(z))))} \tag{1.55}$$

$$B^{*1}(z,0,0) = \frac{aP^*(z,0)(1 - R^{*1}(w_x(z)))}{w_x(z)} \tag{1.56}$$

$$B_i^{*1}(z,0,0) = \frac{a_i P_i^*(z,0)(1 - R_i^{*1}(w_x(z)))}{w_x(z)} \tag{1.57}$$

To derive the total PGF of the system size distribution the following generating functions are considered.

$P_I(z)$ = Partial generating function(PGF) of the system size when the server is in idle state

$$= Q^*(z,0) + P_0(z) + D^*(z,0) = \bar{P}(0)I_N^{SR}(z) \quad \text{[from equations 1.51 to 1.53]}$$

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$P_{comp}(z)$ = The PGF of the system size when the server is in busy or in breakdown state (also called completion state)

$$= P^*(z,0) + \sum_{i=1}^c P_i^*(z,0) + B^*(z,0,0) + \sum_{i=1}^c B_i^{**1}(z,0,0)$$

By adding equations (1.54) to (1.57) as
$$P_{comp}(z) = \frac{zI_N^{SR}(z)\bar{P}(0)}{z - H_{SOS}^*(h(w_x(z)))} [H_{SOS}^*(h(w_x(z))) - 1]$$

Thus the total PGF of the system size distribution is given by

$$P_{SOS(N)}^{SR}(z) = P_1(z) + P_{comp}(z) = \bar{P}(0)I_N^{SR}(z) \frac{(z-1)H_{SOS}^*(w_x(z))}{(z - H_{SOS}^*(w_x(z)))} \tag{1.58}$$

The constant $\bar{P}(0)$ can be calculated by using the normalizing condition $P_{SOS(N)}^{SR}(1) = 1$ and found to be

$$\bar{P}(0) = \frac{1-\rho}{I_N^{SR}(1)}, \text{ where } I_N^{SR}(1) = E(D) + E(V) \sum_{n=0}^{N-1} \frac{\beta_n}{1-\alpha_0} + \sum_{n=0}^{N-1} \frac{\psi_n}{\lambda}, \text{ and } \rho = \lambda E(X)E(H) \text{ where}$$

$$E(H) = E(S)(1 + aE(R)) + \sum_{i=1}^c r_i E(S_i)[1 + a_i E(R_i)]$$

By substituting $\bar{P}(0)$ in the equation(1.58), the total PGF of the model can be written as

$$P_{SOS(N)}^{SR}(z) = \frac{(1-\rho)(z-1)H_{SOS}^*(h(w_x(z))) I_N^{SR}(z)}{(z - H_{SOS}^*(w_x(z))) I_N^{SR}(1)}$$

IV. PERFORMANCE MEASURES

Let P_v, P_{set}, P_{build} and P_{busy} denote the probability that the server is on vacation, setupstate, build up state and in busy state respectively. Then the steady state system size probabilities can be obtained by considering the corresponding equations(1.51 to 1.55) at $z=1$, are given by

(i) $P_v = \bar{P}(0)E(V) \sum_{n=0}^{N-1} \frac{\beta_n}{(1-\alpha_0)}$ (ii) $P_{set} = \bar{P}(0)E(D)$

(iii) $P_{build} = \bar{P}(0) \sum_{n=0}^{N-1} \frac{\psi_n}{\lambda}$ (iv) $P_{busy} = \frac{\bar{P}(0)\lambda E(X)I_N^{SR}(1)}{(1-\rho)} \left[E(S) + \sum_{i=1}^c r_i E(S_i) \right]$

Expected System Size: Let L_v, L_{set}, L_{build} and L_{busy} denote the expected system size when the server is in vacation state, setupstate, build up state and in busy state respectively.

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Then mean system size of the model $L = \frac{d}{dz} (P_{SOS(N)}^{SR}(z))$ at $z=1$ ie, $L = \rho + L_{M^x/G/1} + \frac{I_N^{SR'}(1)}{I_N^{SR}(1)}$

where $L_{M^x/G/1} = \frac{\lambda E(X(X-1))E(H) + (\lambda E(X))^2 E(H^2)}{2(1-\rho)}$ and

$$E(H^2) = E(S^2)(1 + aE(R))^2 + aE(S)E(R^2) + 2E(S)(1 + aE(R))\left(\sum_{i=1}^c r_i E(S_i)(1 + a_i E(R_i))\right) \\ + \sum_{i=1}^c r_i E(S_i^2)(1 + a_i E(R_i))^2 + \sum_{i=1}^c r_i a_i E(S_i) E(R_i^2)$$

and

$$I_N^{SR'}(1) = \lambda E(X) \left[\frac{E(V^2)}{2} + E(V)E(D) \sum_{n=0}^{N-1} \frac{\beta_n}{1-\alpha_0} + \frac{E(D^2)}{2} + \sum_{n=0}^{N-1} \frac{\psi_n}{\lambda} E(D) \right] + E(V) \sum_{n=0}^{N-1} \frac{n\beta_n}{1-\alpha_0} + \sum_{n=0}^{N-1} \frac{n\psi_n}{\lambda}$$

By differentiating the suitable partial generating functions the mean system size correspondingly obtained at $z=1$ are listed below

$$(i) L_v = \left[\frac{\lambda E(X)E(V^2)}{2} \sum_{n=0}^{N-1} \frac{\beta_n}{(1-\alpha_0)} + \sum_{n=0}^{N-1} \frac{n\beta_n}{(1-\alpha_0)} E(V) \right] \bar{P}(0)$$

$$(ii) L_{set} = \bar{P}(0) \lambda E(X) \left[E(D) \left[E(V) \sum_{n=0}^{N-1} \frac{\beta_n}{(1-\alpha_0)} + \sum_{n=0}^{N-1} \frac{\psi_n}{\lambda} \right] + \frac{E(D^2)}{2} \right]$$

$$(iii) L_{build} = \bar{P}(0) \sum_{n=0}^{N-1} \frac{n\psi_n}{\lambda}$$

V. CONCLUSION

It is verified that the model under consideration is the most generalization of N policy queueing models with server single and multiple vacation. It is noted that $L=1$ and $L \rightarrow \infty$ the results of the proposed model coincide with the corresponding results of the classical vacation N policy queueing models of Lee et al.(1995) and Lee et al.(1994) respectively.

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